Dimensional Reasoning & Dimensional Consistency Testing

Nathaniel Osgood

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Talk Outline

- Motivations
- Dimensional Systems
- Dimensional Analysis
- Examples
- Discussion

Motivations

- General
  - Dimensional analysis (DA) critical historically for
    - Scoping models
    - Formulating models
    - Validating models
    - Calibrating models
  - Systems modeling community has made important but limited use of DA
  - Strong advantages from & opportunities for improved DA use
- Specific
  - Performance concerns for public health models

Dimensions and Units

- *Dimensions* describe semantic category of referent
  - e.g. Length/Weight/Pressure/Acceleration/etc.
  - Describe referent
  - Independent of size (or existence of) measure
  - No conversions typical between dimensions
  - A given quantity has a unique dimension
- *Units* describe references used in performing a particular measurement
  - e.g. Time: μSeconds/Weeks/Centuries
  - This is *metadata*: Describes measured value
  - Relates to a *particular dimension*
  - Describe measurement of referent
  - Dimensional constants apply between units
  - A given quantity can be expressed using many units
  - Even dimensionless quantities can have units

Units & Dimensions

- Frequency
  - Dimension:1/Time
  - Units: 1/Year, 1/sec, etc.
- Angle
  - Dimension: “Dimensionless” (1, “Unit”)
  - Units: Radians, Degrees, etc.
- Distance
  - Dimension: Length
  - Units: Meters/Fathoms/Li/Parsecs

Dimensional Homogeneity: Distinctions

- Adding items of different dimensions is semantically incoherent
  - Fatally flawed reasoning
- Adding items of different units but the same dimension *is* semantically sensible but numerically incorrect
  - Requires a conversion factor

Structure of Dimensional Quantities

- Dimensional quantity can be thought of as a pair

  (value, m) where value$\in\Re$ and m$\in\Re^d$

- Quantity's dimension/units can be represented as
  - Products of powers of "reference" dimensions/units

    Rate of water flow: L^3T^{-1}
  - Vectors in a *d* dimensional vector space (of ref. dimens.)
    - Each index in the vector represents the exponent for that reference dimension/unit
- Dimension dictates the value scaling needed for *unit conversion*
  - A *dimensionless* quantity holds the same value regardless of measurement system
- Dimensional quantities have operations that are related to but more restricted than for e.g. $\Re$

A Particularly Interesting Dimensionality: “Unit” Dimension

- Recall:dimensions associated with quantities can be expressed as “product of powers”
- We term quantities whose exponents are all 0 as being of “unit dimension”
- Another term widely used for this is “Dimensionless”
  - This is somewhat of a misnomer, in that these quantities do have a dimension – just a very special one
    - Analogy: calling something of length 0 “lengthless”
- Such quantities are independent of unit choice

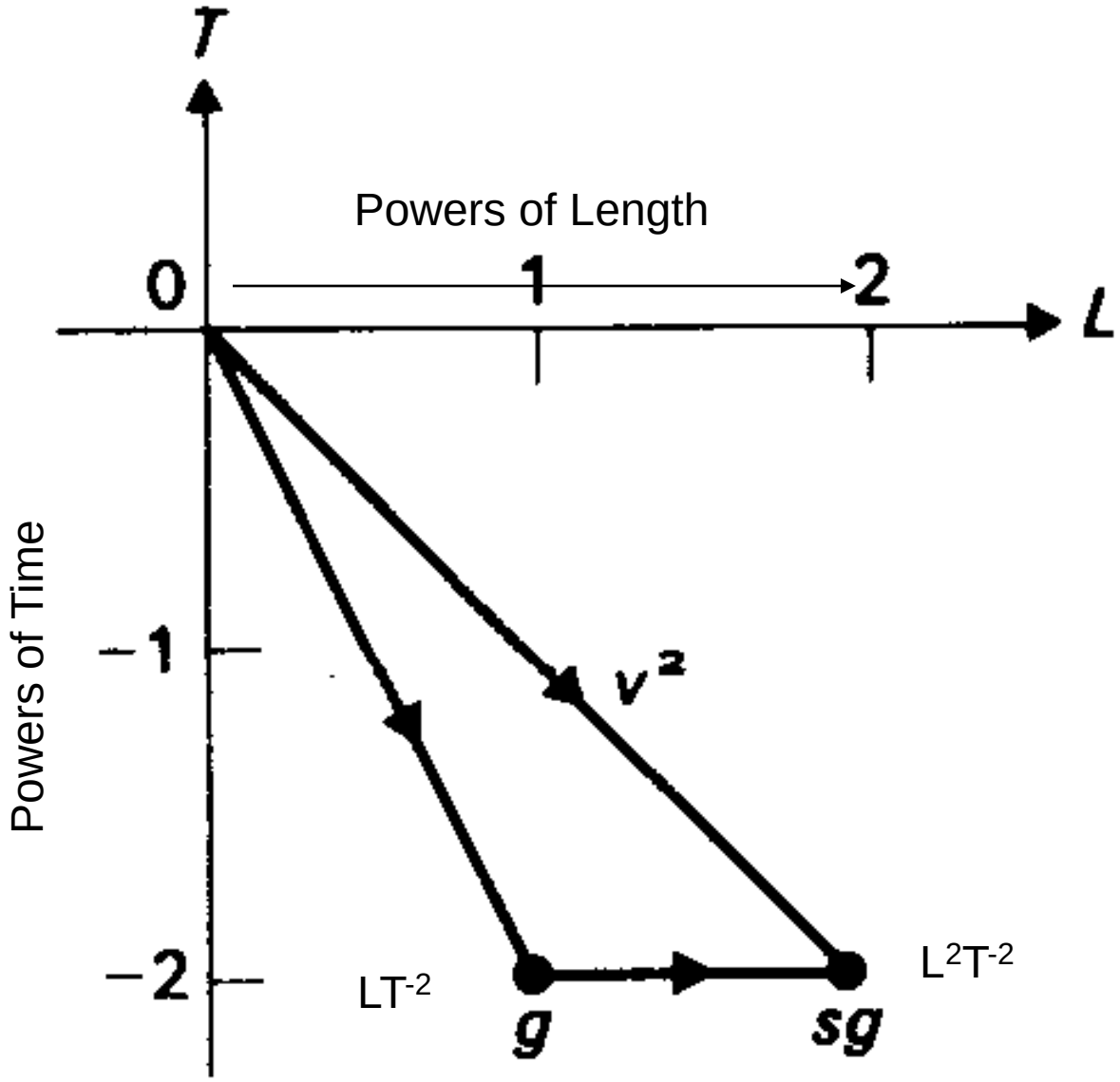
Dimensionality & Unit Choice

- Exponent for dimension dictates the numerical value scaling required by *unit conversion*
  - Consider x=1 \$/ft and y=1 \$/ft^2
    - Consider converting from feet to meters
      - x=1 \$/ft * (1ft/1m) ≈ 3.208 \$/m
      - y= 1 \$/ft^2 * (1ft/1m)^2 ≈ 10.764 \$/m^2
- ***A dimensionless quantity maintains the same numeric value regardless of measurement system***
  - Cf: Fraction = .1 (Unit Dimension)
  - 100 ft^2/1000 ft^2 =.1

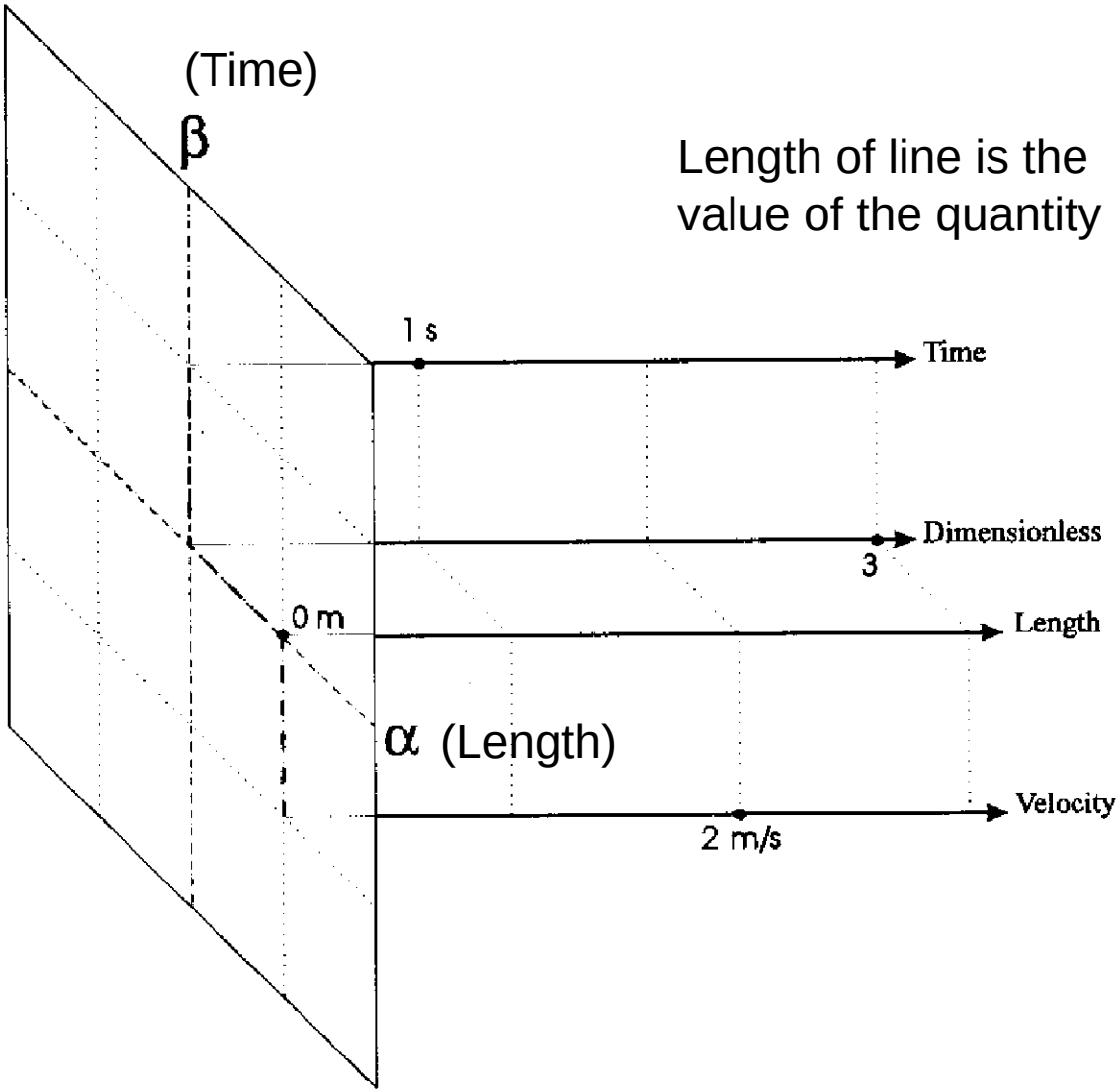
Common Quantities of Unit Dimension

- Fractions of some quantity
- Likelihoods (probabilities)

Dimensional Space



Quantities in Dimension Space



Treating all quantities as dimensionless loses information (projects purely onto the z dimension)

Stock-Flow Dimensional Consistency

- Invariant: Consider a stock and its inflows and outflows. For any flow, we must have

  [Flow]=[Stock]/Time

- This follows because the Stock is the integral of the flow
  - Computing this integral involves summing up many timesteps in which the value being summed is the flow multiplied by time.

Seeking Hints as to the Dimension Associated w/a Quantity

- How is it computed in practice?
  - What steps does one go through to calculate this? Going through those steps with dimensions may yield a dimension for the quantity
- Would its value need to be changed if we were to change diff units (e.g. measure time in days vs. years)?
- Is there another value to which it is converted by some combination with other values?
  - If so, can leverage knowledge of dimensions of those other quantities

Computing with Dimensional Quantities

- To compute the dimension (units) associated with a quantity, perform same operations as on numeric quantities, but using dimensions (units)
- We are carrying out the same operations in parallel in the numerics and in the dimensions (units).
  - With each operation, we can perform it twice
    - Once on the numerical values
    - Once on the associated dimensions

Dimensional Homogeneity

- There are certain computations that are dimensionally inconsistent are therefore meaningless
- Key principle: Adding together two quantities whose dimensions differ is dimensionally "inhomogeneous" (inconsistent) & meaningless
- By extension

  a^b is only meaningful if b is dimensionless

  Derivation: $a^b = ((a/e)e)^b = (a/e)^b e^b = (a/e)^b(1+b+b^2/2 +b^3/3*2*1\ldots)$

  The expression on the right is only meaningful if [b]:1

Dimensional Notation

- Within this presentation, we'll use the notation

  [x]: D to indicate quantity x is associated with dimension D

- For example,

  [x]: $

  [y]: Person/Time

  [z]: 1

Example

$$\frac{a+(b*c)}{d}$$

Suppose further that

[a]: Person

[b]: Person/Time

[c]: Time

[d]: $

To compute the dimensions, we proceed from “inside out”, just as when computing value

- [b*c]=[b]*[c]=

(Person/Time)*Time=Person

- [a+(b*c)]=[a]+[b*c]=Person +Person=Person
- Thus, the entire expression has dimension

  [a+(b*c)/d] = [a+(b*c)/d]/[d] =Person/$

Lotka Volterra model

- Variables Dimensions

  [β]: 1/(Fox * Time)

  [γ]: 1/(Hare * Time)

  [δ],[α]: 1/Time

$$\dot{H} = -\beta HF + \alpha H$$

$$\dot{F} = \gamma HF - \delta F$$

- Cf: Frequency of oscillations: [λ] : (1/Time)
  - Clearly cannot depend on β or γ, because
    - These parameters would introduce other dimensions
    - Those dimensions could not be cancelled by any other var.
- The exponent of Time in [λ] is -1
- By symmetry, the period must depend on both α and δ, which suggests $\sqrt{\alpha\delta}$

Classic SIR model

$$\dot{S} = -cS\left(\frac{I}{S+I+R}\right)\beta$$

$$\dot{I} = cS\left(\frac{I}{S+I+R}\right)\beta - \frac{I}{\mu}$$

$$\dot{R} = \frac{I}{\mu}$$

- Variables Dimensions

  [S]=[I]=[R]: Person

  [β]: 1 (A likelihood!)

  [c]: (Person/Time)/Person=1/Time

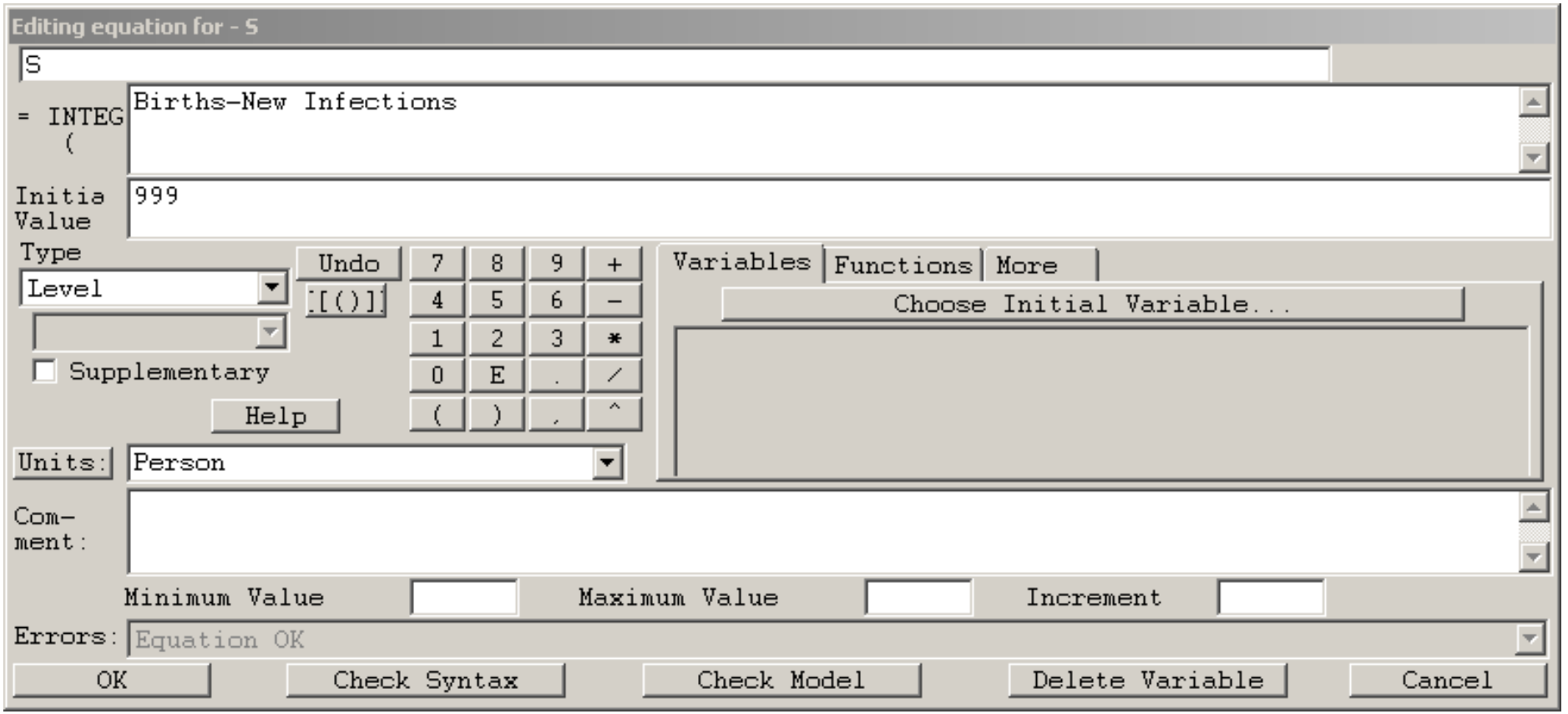
  (Just as could be calculated from data on contacts by *n* people over some time interval)

  [μ]: Time

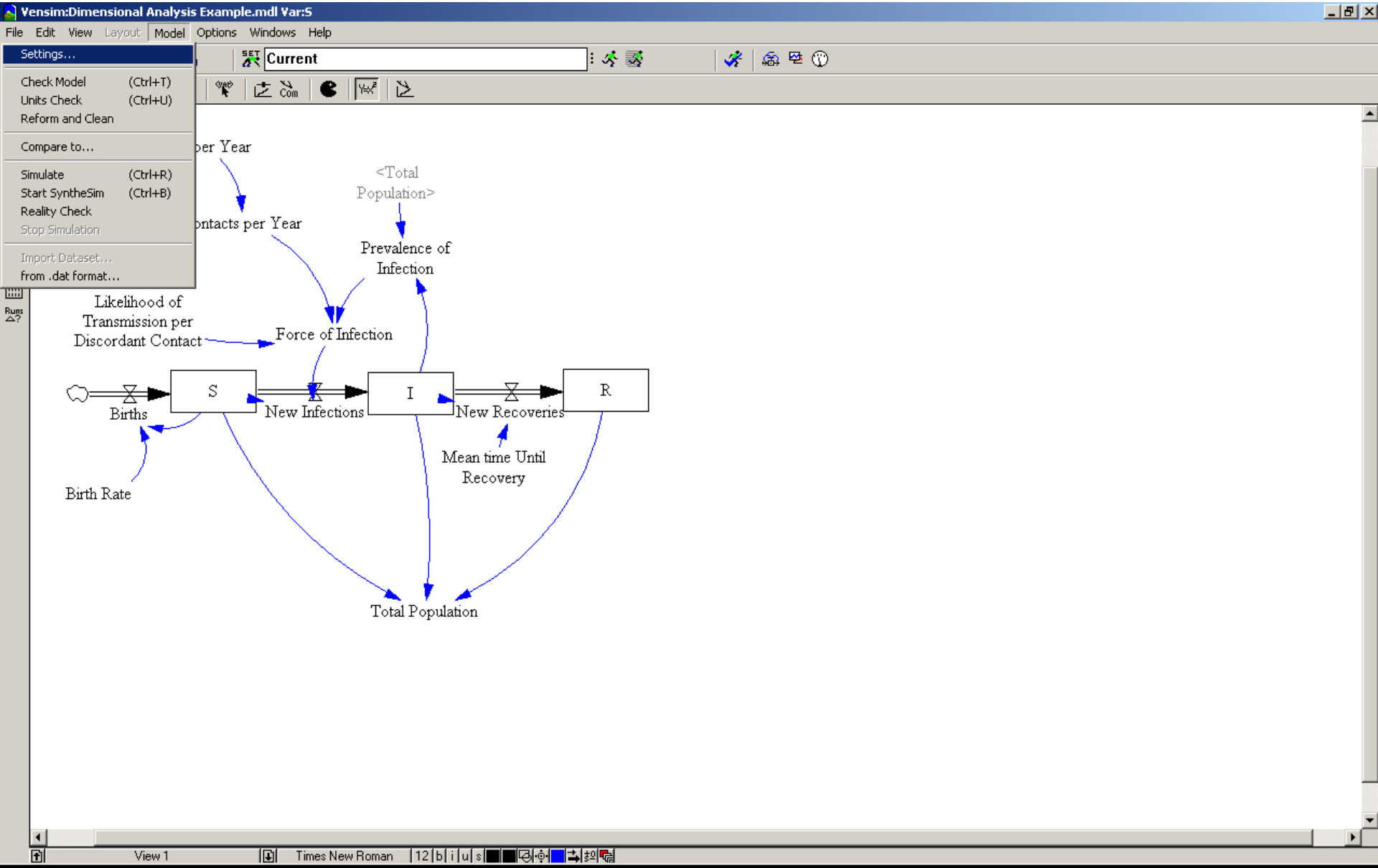
Note that the force of infection $\lambda = c\left(\frac{I}{S+I+R}\right)\beta$ has units 1/Time, which makes sense

  - Firstly, multiplying it by S must give rate of flow, which is Person/Time
  - Secondly, the reciprocal of such a transition hazard is just a mean duration in the stock, which is a Time => dimension must be 1/Time

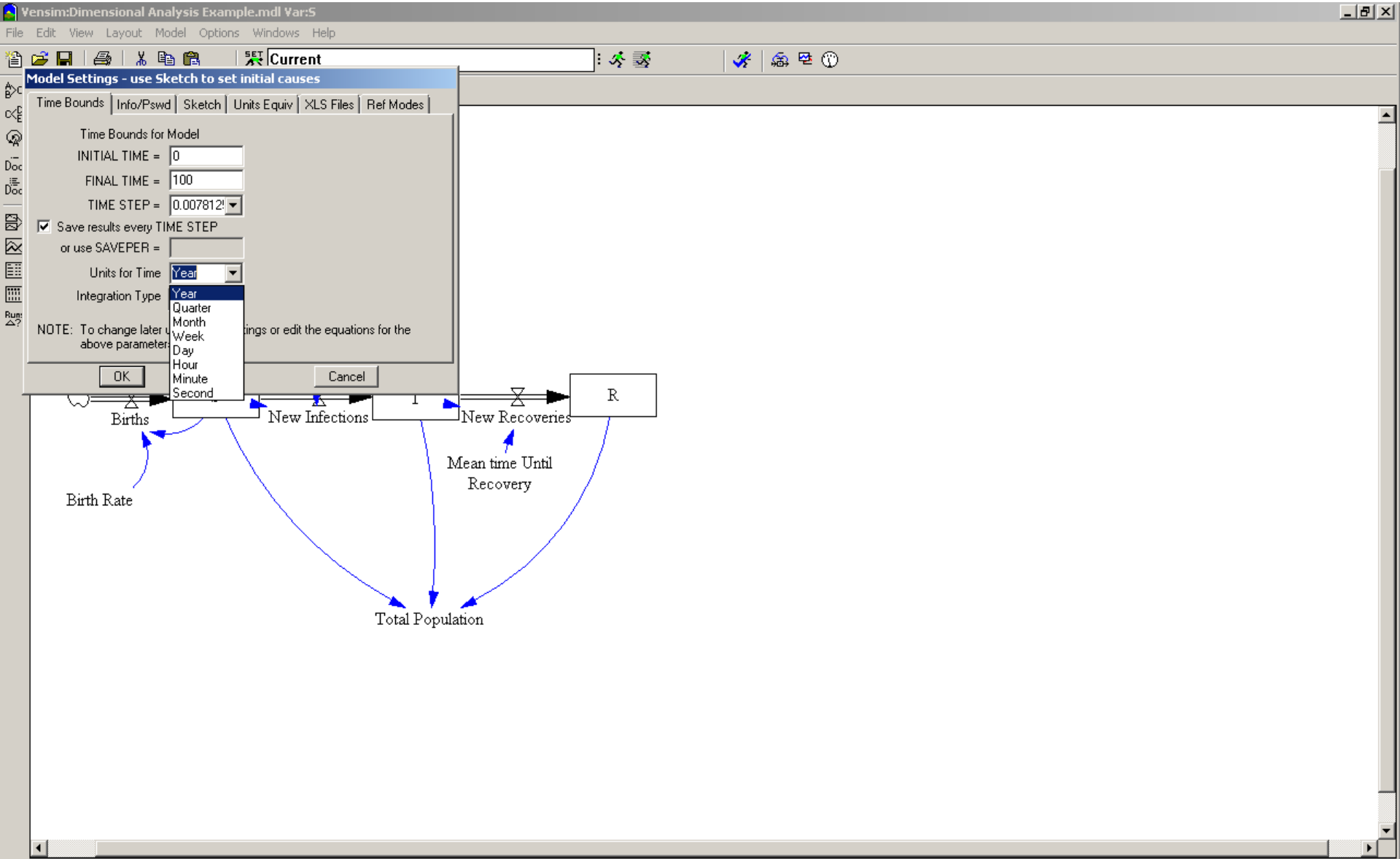
Indicating Units Associated with a Variable in Vensim



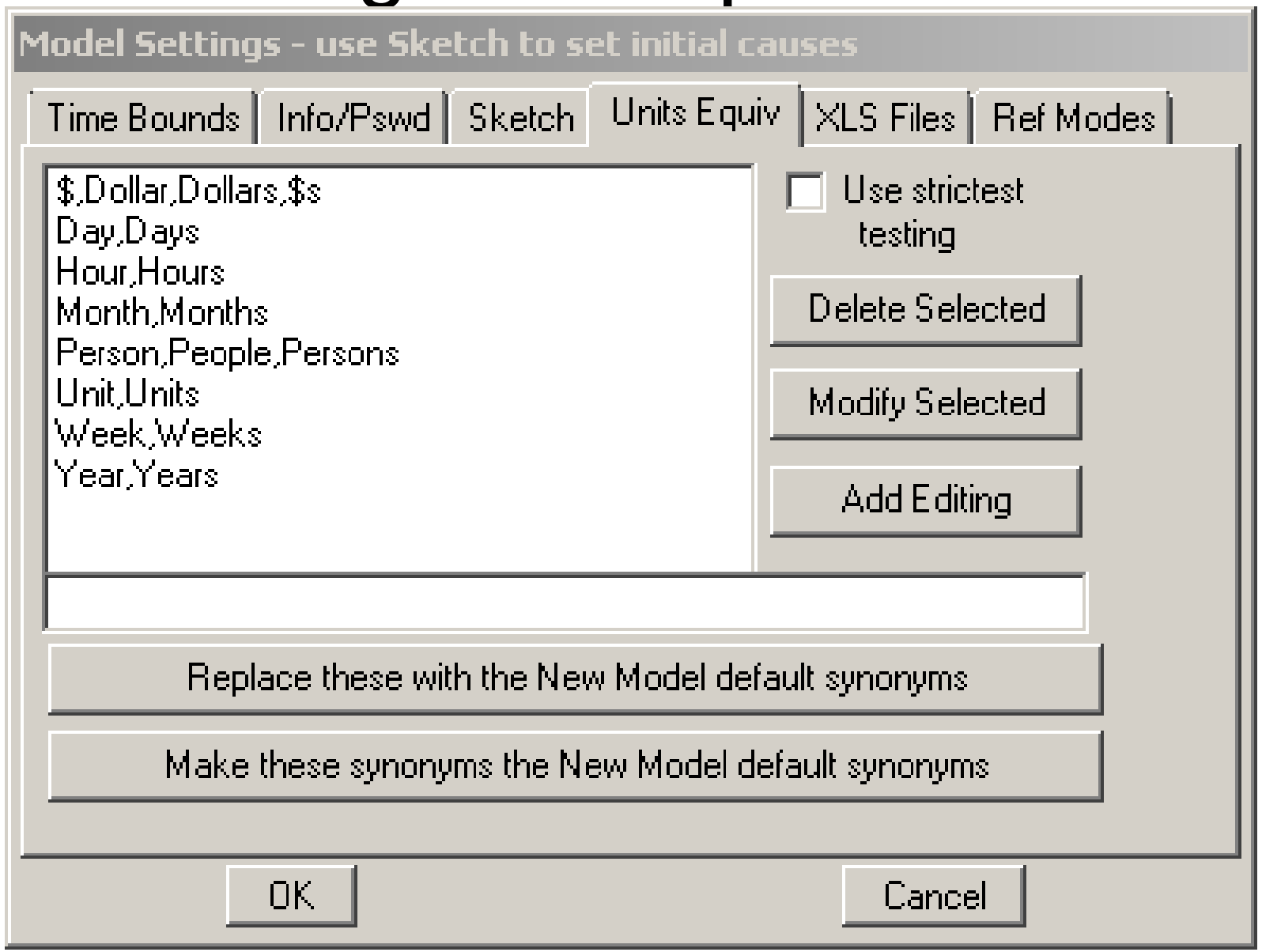
Accessing Model Settings



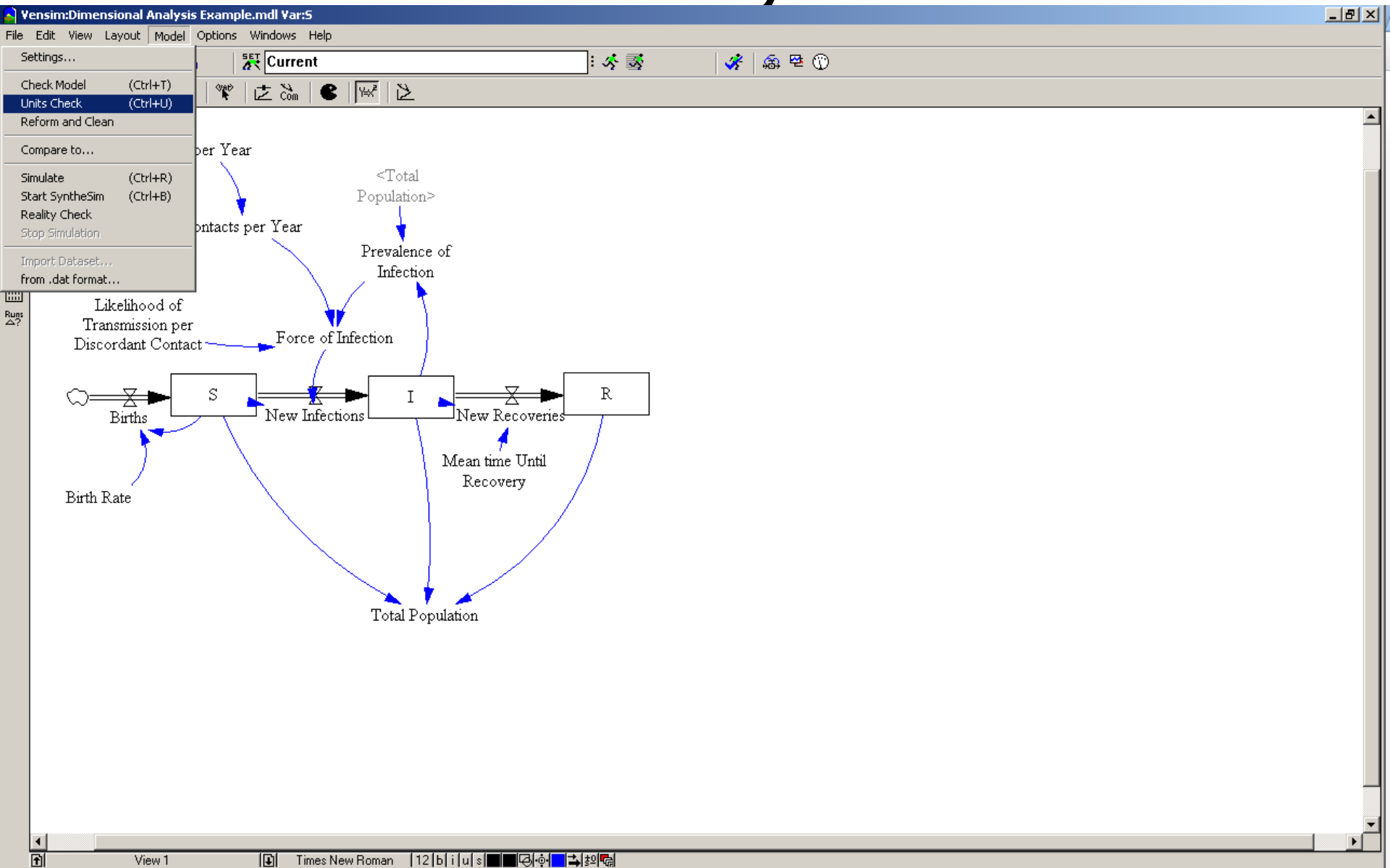
Choosing Model Time Units



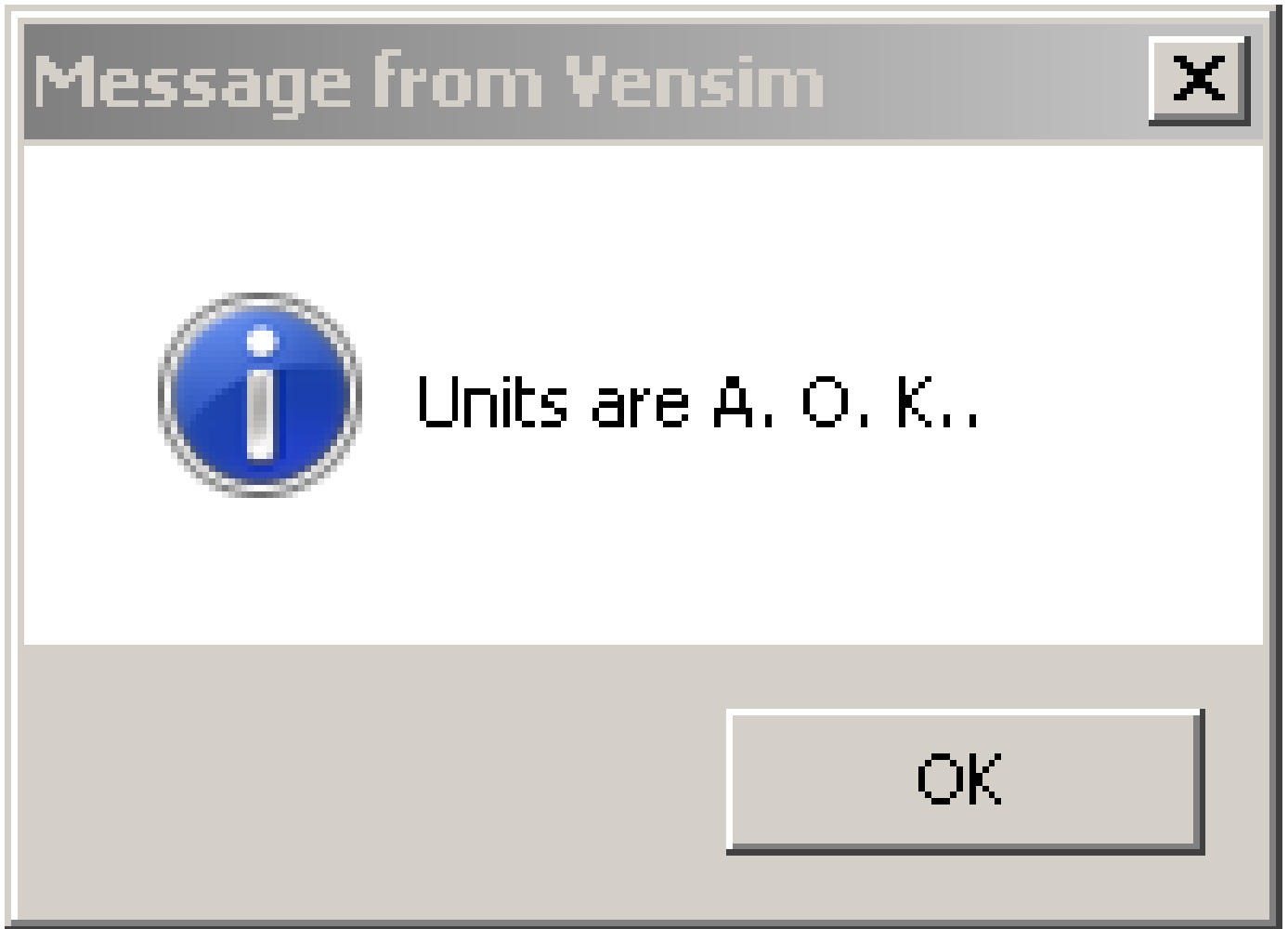
Setting Unit Equivalence



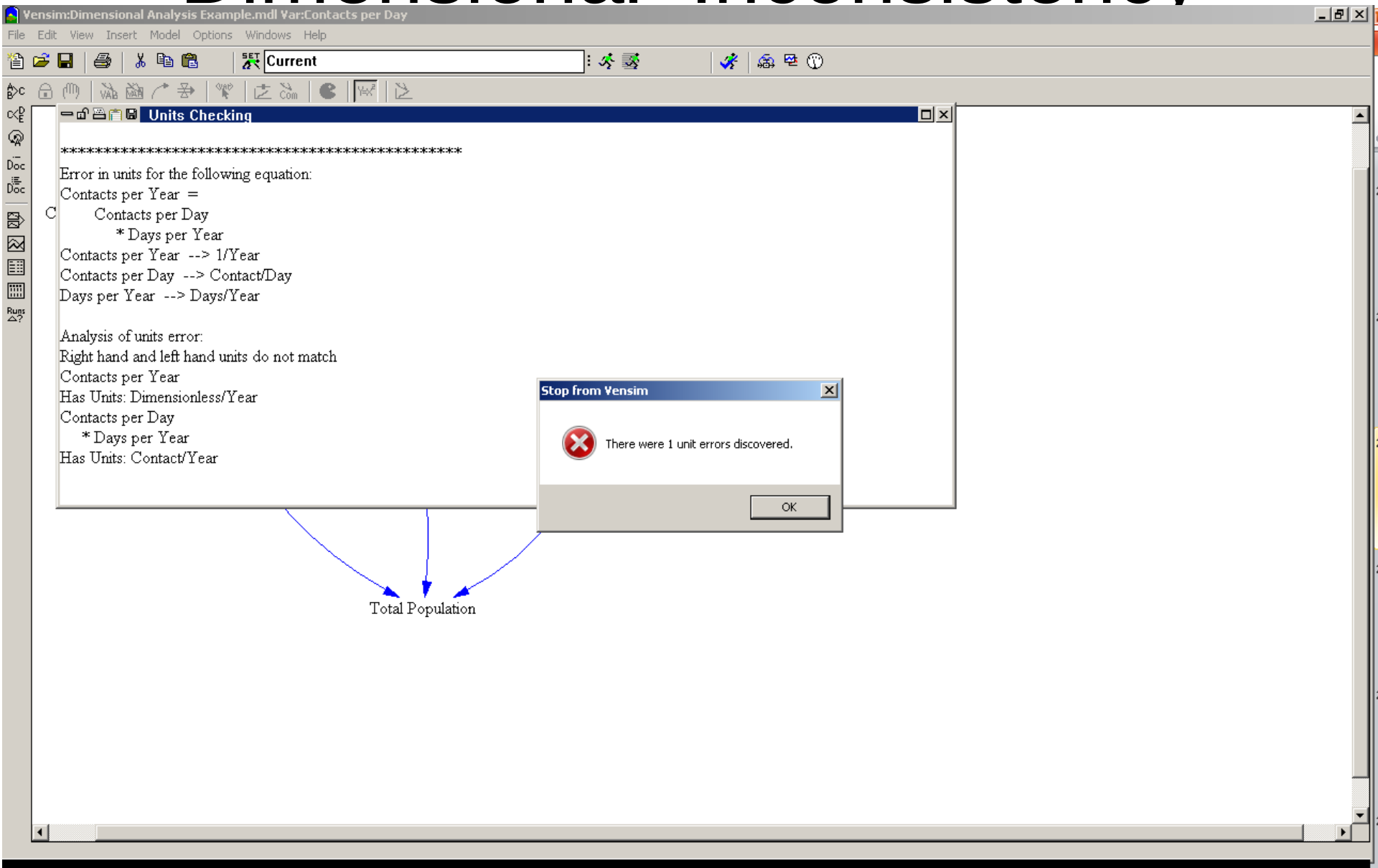
Requesting a Dimensional Consistency Check



Confirmation of Unit Consistency



Indication of (Likely) Dimensional Inconsistency



Vensim Interface

- Vensim will perform dimensional simplification via simple algebra on dimensional expressions
  - E.g. Person/Person is reduced to 1
- In some vensim modes, when the mouse hovers over a variable, Vensim will show a pop-up “tab tip” that shows the dimension for that variable
- Vensim can check many aspects of dimensional consistency of a model

Vensim Capabilities

- Associate variables with units
- Define new units (beyond built-in units)

  e.g. Person, Deer, Bird, Capsule
- Define unit equivalence

  e.g. “Day”, “Days”